1. Evaluate the Fourier-series solution of the initial-boundary-value problem for the 1-D wave equation

$$u_{tt} = 9\,u_{xx}\,, \quad \text{for} \;\; 0 < x < \pi, \;\; t > 0, \tag{1}$$

$$u(x,0) = \sin(5x), \;\; u_t(x,0) = 1, \;\; \text{for} \;\; 0 < x < \pi, \;\; u(0,t) = u(\pi,t) = 0 \;\; \text{for} \;\; t > 0,$$

as obtained by separation of variables.

2. Apply the method of separation of variables to obtain a formal Fourier-series solution of the telegraph equation

$$u_{tt} + A\,u_t + B\,u = c^2 u_{xx}\,, \quad \text{for} \;\; 0 < x < L, \;\; t > 0, \tag{2}$$

with boundary and initial conditions

$$u(0,t) = u(L,t) = 0 \;\; \text{for} \;\; t > 0, \;\; u(x,0) = \varphi(x), \;\; u_t(x,0) = 0 \;\; \text{for} \;\; 0 < x < L, \tag{3}$$

where A, B, and c are positive constants with $A^2L^2 < 4(BL^2 + c^2\pi^2)$.

3. Solve the following problem for the diffusion equation (k and T are given positive constants):

$$u_t = k\,u_{xx}\,, \;\; u(0,t) = u(3,t) = T\,, \;\; u(x,0) = T\,, \;\; \text{for} \;\; 0 \le x \le 3, \; t \ge 0. \tag{4}$$

Give a physical interpretation of your solution in terms of heat diffusion.

4. Assume that the function u is a solution of the diffusion equation $u_t = k\,u_{xx}$ for $0 < x < L, \; t > 0$.

   (a) Show that u satisfies the integral expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}u^2(x,t)\,\mathrm{d}x = k\,u(x,t)\,u_x(x,t)\Big|_{x=0}^{x=L} - k\int_0^L u_x^2(x,t)\,\mathrm{d}x \tag{5}$$

   (start with multiplying the heat equation by u, then integrate over $[0, L]$).

   (b) Show that the initial-boundary-value problem for the heat equation

$$u_t = k\,u_{xx}\,, \;\; u(0,t) = u(L,t) = 0\,, \;\; u(x,0) = f(x)\,, \;\; \text{for} \;\; 0 \le x \le L, \;\; t \ge 0, \tag{6}$$

   with f a given function, can have only one solution.
   Procedure: Assume there are two solutions; take the difference and determine the conditions (PDE & boundary / initial values) that are satisfied by that function. Evaluate the result above for the difference function.

5. Consider the partial differential equation

$$u_t = k\,(u_{xx} + a\,u_x + b\,u) \tag{7}$$

with constants k, a, b and $k > 0$. Using a transformation of the form

$$u(x,t) = \mathrm{e}^{\alpha x + \beta t} w(x,t)\,, \tag{8}$$

determine constants α and β, such that the function w satisfies the standard heat equation $w_t = k\,w_{xx}$.

Good luck !