1. Consider a thin homogeneous bar of given length with uniform cross section, whose sides are well insulated. The right end is kept at a temperature that is externally prescribed, but varies with time. The left end of the bar is insulated. At a specific time the temperature distribution in the bar is known. Formulate an initial-boundary value problem that models the diffusion of heat through this bar.

2. Suppose u is a bounded solution of the initial value problem

$$u_t = K\,u_{xx} \;\text{ for }\; -\infty < x < \infty,\;\; t > 0, \quad u(x,0) = f(x) \;\text{ for }\; -\infty < x < \infty \tag{1}$$

for a given continuous function f, with $u(x,t) \to 0$ uniformly in t as $x \to \pm\infty$. K is a positive constant. Show that

$$|u(x,t)| \le \max_{x\in\mathbb{R}} |f(x)| \tag{2}$$

for all x, t with $-\infty < x < \infty$ and $t > 0$.
(Apply the weak maximum principle on an interval $x \in [-a, a]$ for $t \ge 0$, then take the limit $a \to \infty$).

3. Consider the initial-boundary value problem for the heat equation on a finite interval

$$u_t = 2\,u_{xx}\,, \;\text{ for }\; 0 < x < 1,\;\; t > 0, \tag{3}$$

with homogeneous boundary conditions $u(0,t) = u(1,t) = 0$ for $t > 0$,
and initial condition $u(x,0) = \sin(3\pi x)$.

   (a) Evaluate the formal general solution as found by separation of variables for this specific problem.
   (b) Show explicitly that the function satisfies the initial-boundary value problem.
   (c) Draw qualitative graphs of the solution at times $t = 0$ and $t = 1/(18\pi^2)$.

4. Recall the treatment of the initial value problem for the heat equation on the entire spatial axis

$$u_t = K\,u_{xx}\,, \quad \text{for} \quad -\infty < x < \infty,\ \ t > 0,\ \ u(x,0) = f(x) \ \text{ for } \ -\infty < x < \infty, \tag{4}$$

where f is a given function, K a positive constant. As detailed during the lecture, by separation of variables one arrives at the formal solution

$$u(x,t) = \int_0^\infty \{a_k \cos(kx) + b_k \sin(kx)\}\ \mathrm{e}^{-k^2 K t} \mathrm{d}k\,, \tag{5}$$

with $\quad a_k = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\cos(kx)\,\mathrm{d}x\,, \quad b_k = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)\sin(kx)\,\mathrm{d}x\,.$

Show that the solution (5) can be given the following alternative forms:

(a) $\quad u(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-k^2Kt}\cos(k(x-\xi))\,\mathrm{d}\xi\,\mathrm{d}k\,,$

(b) $\quad u(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}(k)\,\mathrm{e}^{-k^2Kt}\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \quad \text{with} \quad \hat{f}(k) = \int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-\mathrm{i}k\xi}\mathrm{d}\xi\,,$

(c) $\quad u(x,t) = \dfrac{1}{2\sqrt{\pi K t}}\displaystyle\int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-\frac{(x-\xi)^2}{4Kt}}\,\mathrm{d}\xi\,,$

(d) $\quad u(x,t) = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} f(x + 2z\sqrt{Kt})\,\mathrm{e}^{-z^2}\,\mathrm{d}z\,.$

Hints:
(4a) — start with the new expression, apply suitable manipulations to arrive at equation (5);
(4b) — manipulate the expression with the aim to reach the form of (4a);
(4c) — start e.g. with (4a), try to reach (4c) by using the rule for the Fourier-transformation of $\exp(-y^2)$:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\cos(\alpha y)\,\mathrm{d}y = \sqrt{\pi}\,\mathrm{e}^{-\alpha^2/4}\,;$$

(4d) — is already quite close to (4c).

5. Consider again the different forms of the formal solution of the initial value problem (4) for the diffusion equation on the entire spatial axis, as derived in exercise 4.

   (a) Verify that the solution satisfies the initial condition.

   (b) Show that a specific symmetry of f transfers to the solution, i.e. show that the solution is even with respect to the spatial variable, if f is an even function, and that the solution is odd, in case f is an odd function.

   (c) Suppose that f is non-negative, that f is limited to a finite spatial interval, $f(x) = 0$ for $|x| > a$, and that f is nonzero in a region around the origin, $f(x) > \epsilon$ for $|x| < b$, where $0 < b < a$ and $\epsilon > 0$. Show that $u(x,t) > 0$ for $-\infty < x < \infty$, for all $t > 0$.

   (Interpretation of this conclusion: The parabolic diffusion equation propagates disturbances with "infinite velocity"; compare with the propagation of disturbances along the edges of the characteristic triangle in case of the hyperbolic wave equation).

   (d) Evaluate the Fourier-integral solution for the specific case of $f(x) = \mathrm{e}^{-4|x|}$.
   (You should obtain a single-integral expression).

Good luck !