1. Consider the Cauchy problem for the partial differential equation

$$u_{xx} + 2x\, u_{xy} + u_{yy} - u_x + y^2\, u = 0 \tag{1}$$

with Cauchy data given on the x-axis: $u(x,0) = x(1-x)\,,\ u_y(x,0) = \cos(2x)\,.$

Verify that the x-axis is not a characteristic of equation (1). Then use the Cauchy data and the differential equation to compute the first four terms of the Taylor-expansion (i.e. terms with up to third order derivatives with respect to y) of the solution of the Cauchy problem around the x-axis.

2. Consider a 1-D wave equation on the entire spatial axis

$$u_{tt} = 9u_{xx}\,, \quad -\infty < x < \infty\,, \quad t > 0\,, \tag{2}$$

with initial data

$$u_t(x,0) = 0\,, \quad u(x,0) = \begin{cases} \sin^2(\pi x) & \text{for } -2 < x < 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Write the solution of the Cauchy problem as a sum of forward and backward waves. Based on that splitting, graph (qualitatively) the solution at times $t = 0,\ t = 1/6,\ t = 1/3$, and $t = 2/3$.

3. Given the 1-D wave equation on the positive real axis

$$u_{xx} - \frac{1}{4}\, u_{tt} = 0 \quad 0 < x < \infty\,, \quad t > 0\,, \tag{4}$$

the initial data $u(x,0) = x\,\sin(\pi x)\ ,\ u_t(x,0) = x(1-x)\,,$ for $x > 0\,,$
and the homogeneous boundary condition $u(0,t) = 0\,,$
evaluate the (piecewise defined) solution of the initial-boundary value problem. Verify explicitly that your solution satisfies the initial- and boundary conditions, and that the function is continuous.

4. Consider a Cauchy problem for the inhomogeneous 1-D wave equation on the entire spatial axis

$$u_{tt} = c^2\, u_{xx} + P(x,t) \text{ for } -\infty < x < \infty,\ t > 0, \tag{5}$$

with initial conditions $u(x,0) = \varphi(x)\,,\ u_t(x,0) = \psi(x)$, for given functions P, φ, and ψ.

(a) Show that the function

$$u(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s + \frac{1}{2c}\int_{\Delta(x,t)} P(x',t')\,\mathrm{d}A' \tag{6}$$

solves the Cauchy problem. Here the last term indicates an integral over the characteristic triangle Δ associated with the point (x,t).

(b) Evaluate the solution (6) for the wave equation with $c = 3$ and inhomogeneity $P(x,t) = 3xt$, with initial data $\varphi(x) = 3\sin(x)$ and $\psi(x) = x$. Check explicitly that your function satisfies the inhomogeneous wave equation (5) and the initial conditions.

Solutions are to be handed in at the beginning of the lecture on December 12, 2005, at 15:45 in the RA "Studio".
Good luck !