1. Consider a 1-D wave equation with a forcing term and homogeneous initial and boundary conditions:

$$u_{tt} = 4\,u_{xx} - \sin(5x)\,, \quad \text{for} \;\; 0 < x < \pi,\; t > 0, \tag{1}$$

$$u(x,0) = u_t(x,0) = 0\,, \quad \text{for} \;\; 0 \le x \le \pi, \quad \text{and} \;\; u(0,t) = u(\pi,t) = 0\,, \quad \text{for} \;\; t > 0.$$

   (a) By applying an ansatz of the form

$$u(x,t) = U(x,t) + f(x) \tag{2}$$

   with a suitably chosen function f, transform (1) into a problem for U, where U satisfies a homogeneous wave equation with inhomogeneous initial conditions.

   (b) Solve the problem for U by separation of variables.

   (c) Show explicitly that the expression for u obtained in this way satisfies all conditions of problem (1).

2. Assume that u satisfies an inhomogeneous 1-D wave equation on an interval $[0, L]$

$$u_{tt} = c^2 u_{xx} + f(x,t)\,, \quad \text{for} \;\; 0 < x < L,\; t > 0, \tag{3}$$

with given inhomogeneity f and constant phase velocity $c > 0$. Show that the following integral relationship holds for any interval $[a, b]$ with $0 < a, b < L$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \frac{1}{2} \left(u_t^2 + c^2\, u_x^2\right) \,\mathrm{d}x = c^2\, u_t\, u_x \Big|_a^b + \int_a^b f\, u_t \,\mathrm{d}x\,. \tag{4}$$

Hint: Multiply the wave equation by u_t, integrate over $[a, b]$, then use integration by parts on a suitable term. Note that $\partial_t(u_t^2) = 2\, u_t\, u_{tt}$.

3. Consider a long, thin, homogeneous bar of length L. Let $u(x,t)$ be the temperature of the bar at position x at time t. We assume that the sides are poorly insulated, such that heat radiates freely away along the sides of the bar. The local change in temperature due to the radiation will be proportional to the difference between the local temperature u of the bar and the (constant) temperature T of the surrounding medium. That leads to the following equation for the temperature distribution in the bar:

$$u_t = k\, u_{xx} - A(u - T) \tag{5}$$

with positive heat diffusion constant k and a positive transfer coefficient A.

   (a) Formulate an initial condition that models a temperature distribution according to a given function f, and boundary conditions that correspond to insulated ends of the bar at $x = 0$ and $x = L$.

   (b) Transform the problem to an initial-boundary-value problem for the standard heat equation (Consider the local temperature difference $w = u - T$; recall classroom exercise no. 5 of week 50).

   (c) Using the Fourier-series solution of the transformed equation, write the formal solution of the original problem for equation (5).

4. Consider the diffusion problem on the positive spatial axis

$$u_t = k\, u_{xx}\,, \quad \text{for} \;\; x > 0,\; t > 0, \tag{6}$$

with initial and boundary conditions

$$u(x,0) = f(x) \;\; \text{for} \;\; x \ge 0, \quad u_x(0,t) = 0 \;\; \text{for} \;\; t \ge 0. \tag{7}$$

Here k is a positive constant, f a given function. Apply separation of variables to obtain a *bounded* formal solution of the problem in terms of Fourier integrals.

Solutions are to be handed in at the beginning of the lecture on January 09, 2006, at 15:45 in the RA "Studio". Good luck !