1. Consider the following mixed boundary-value problem for the Laplace equation in $\mathbb{R}^2$

$$\Delta u(x,y) = 0, \text{ for } 0 < x < a,\ 0 < y < b, \tag{1}$$

with boundary conditions

$$u_y(x,0) = 0,\ u(x,b) = g(x), \text{ for } 0 \le x \le a,\ \ u(0,y) = f(y),\ u(a,y) = 0, \text{ for } 0 \le y \le b,$$

where f and g are given, suitably smooth functions.

   (a) Assume that p is a solution of problem (1) for $g = 0$, and that q satisfies the problem (1) for $f = 0$. Verify that the sum $v = p + q$ is a solution of the full problem (1).

   (b) Apply the method of separation of variables to obtain a solution of problem (1) for the specific boundary data

$$f(y) = 5\cos\left(\frac{11}{2}\frac{\pi}{b}y\right) \text{ and } g = 0\,. \tag{2}$$

   Check explicitly that your solution satisfies all conditions of problem (1).

2. Prove the representation theorem for functions in $\mathbb{R}^2$ by following the procedure that was detailed in the lecture for functions in $\mathbb{R}^3$:

   (a) Derive Green's second identity

$$\iint\limits_{\Omega} (u\Delta v - v\Delta u)\,\mathrm{d}A = \oint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}s\,, \tag{3}$$

   which holds for functions $u, v \in C^2(\Omega)$, where $\Omega \in \mathbb{R}^2$ is a bounded domain with a piecewise smooth, closed curve as its boundary $\partial\Omega$.

   (b) As shown in exercise (1) of this course, the function

$$v(\boldsymbol{y}) = \ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|} \tag{4}$$

   is harmonic in Ω with the exception of point $\boldsymbol{x}$. Apply the identity (3) to a domain formed by removing a ball around $\boldsymbol{x}$ from Ω, with the function (4) inserted for v (note that all integrations are meant with respect to $\boldsymbol{y}$). Then apply manipulations, limits, and simplifications analogous to the $\mathbb{R}^3$-case, until you arrive at the representation theorem for $\mathbb{R}^2$:

$$u(\boldsymbol{x}) = \frac{1}{2\pi}\oint_{\partial\Omega}\left\{\frac{\partial u}{\partial n}(\boldsymbol{y})\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|} - u(\boldsymbol{y})\frac{\partial}{\partial n}\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\right\}\mathrm{d}s - \frac{1}{2\pi}\iint\limits_{\Omega}\Delta u(\boldsymbol{y})\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\mathrm{d}A\,. \tag{5}$$

3. As discussed in the lecture, the Laplace equation for the upper half-plane with boundary conditions on the x-axis

$$u_{xx} + u_{yy} = 0, \;\; u(x,0) = f(x), \;\; x \in \mathbb{R}, \;\; y > 0, \tag{6}$$

has the solution

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\, f(\xi)}{y^2 + (\xi - x)^2} \, \mathrm{d}\xi. \tag{7}$$

   (a) Consider the equation in the upper left quadrant with homogeneous boundary conditions on the positive y-axis:

   $$v_{xx} + v_{yy} = 0, \;\; v(x,0) = g(x), \;\; v(0,y) = 0, \;\; x < 0, \;\; y > 0. \tag{8}$$

   Use the result (7) to find an integral expression for the dependence of the solution of (8) on the function g, given on the negative x-axis.

   (b) Now write a solution for the problem on the lower left quadrant:

   $$w_{xx} + w_{yy} = 0, \;\; w(x,0) = h(x), \;\; w(0,y) = 0, \;\; x < 0, \;\; y < 0. \tag{9}$$

Solutions are to be handed at the beginning of the exercises class on January 18, 2006, at 15:45 in one of the rooms TE 6 or TE 7. Good luck!